MEASURE DECOMPOSITIONS AND APPLICATIONS TO MARKOV CHAIN MIXING

The goal for this note is to compile a few applications of measure decomposition to mixing of Markov chains. To limit the scope of this note, we will consider Glauber dynamics on the vertex of a hypercube $\mathcal{X}_n \coloneqq \{\pm 1\}^n$. The dimension ($n$) we refer to as the number of particles whereas the value (± 1) we will refer to as the *spin* of a particle. We start by defining an Ising model.

Definition 0.1. A probability measure $\mu \in \mathcal{P}(\mathcal{X}_n)$ is the *Ising model* with external field $h \in \mathbb{R}^n$ and interaction $J \in \mathbb{R}^{n\times n}$ is a probability measure defined as

$$\mu(x) = \frac{1}{Z} \exp\left(-\langle x, Jx\rangle + \langle h, x\rangle\right)$$

where $Z = Z(h, J)$ is the normalizing constant (or *partition function*)

$$Z = \sum_{x\in\mathcal{X}_n} \exp\left(-\langle x, Jx\rangle + \langle h, x\rangle\right).$$

Remark 0.2. From now on, we will assume the coupling matrix J is positive definite. This is without loss of generality as self-interaction from the spin always cancels, i.e., $x_i x_i = 1$ for all $x \in \mathcal{X}_n$ and $i \in [n]$. Therefore, we can always add to the diagonal so that J is diagonally dominant when the additional numbers added get absorbed in the normalizing constant.

For every Ising model, we can associate a Markov chain X, taking values in $\mathcal{X}_n$, such that the stationary distribution of X is the Ising model. In fact, there are many Markov chains that satisfies this. We will consider the most canonical one, known as the Glauber dynamics or *Gibbs sampler*.

Definition 0.3. A Markov chain $X = (X_t)_{t\geq 0}$ obeying *Glauber dynamics* is a $\mathcal{X}_n$-valued pure-jump Markov process with generator

$$\mathcal{A}_\mu f(x) = \sum_{i=1}^n \mathbb{E}_\mu[f(X)|X_{-i} = x_{-i}] - f(x) = \sum_{i=1}^n \frac{\mu(x^i)}{\mu(x) + \mu(x^i)}(f(x^i) - f(x))$$

where $x_{-i} \in \{\pm 1\}^n$ denotes the bitstring with entry i removed and x^i denotes flipping the i-th entry. In words, for each index i is resampled according to the conditional measure $\mu(\cdot|x_{-i})$ at a unit rate independent of other indices.

Indeed, by checking for detailed balance under μ:

$$\frac{\mu(x^i)}{\mu(x) + \mu(x^i)}\mu(x) = \frac{\mu(x)}{\mu(x) + \mu(x^i)}\mu(x^i),$$

the desired Ising model μ is indeed invariant and the dynamics is reversible.

Remark 0.4. Some refer to Glauber dynamics more generally as jump processes on the hypercube that are reversible, and refer to the one above as *heat bath* Glauber dynamics. We restrict the scope to this particular choice of dynamics for simplicity and popularity, though other choices of jump rates can simplify some analysis.

Moreover, since $\mu(x) > 0$ for all $x \in \mathcal{X}_n$, the chain is irreducible and will converge to the invariant measure in the limit in the sense that

$$\mathcal{L}(X_t) \to \mu \text{ as } t \to \infty.$$

However, we are interested in the rate of convergence. In particular, we will obtain rates of convergence through establishing functional inequalities, which investigates properties of the Dirichlet form.

Definition 0.5. The Dirichlet form for Glauber dynamics corresponding to Ising model μ is defined to be

$$\mathcal{E}_\mu(f,g) = -\mathbb{E}_\mu[f\mathcal{A}_\mu g] = \frac{1}{2}\sum_{i=1}^{n}\sum_{x\in\mathcal{X}_n}\frac{\mu(x)\mu(x^i)}{\mu(x)+\mu(x^i)}(f(x^i)-f(x))(g(x^i)-g(x))$$

for any $f : \mathcal{X}_n \to \mathbb{R}$. We write $\mathcal{E}_\mu(f) = \mathcal{E}_\mu(f,f)$.

With Dirichlet forms in mind, functional inequalities we want to prove generally follows the form:

> if [famous name] inequality holds, that is
> $$\mathcal{E}_\mu([\text{some function of } f]) \geq c\cdot[\text{functional}]_\mu(f)$$
> for [a large class of] f, then $\mathcal{L}(X_t) \to \mu$ at a rate $\mathcal{O}(e^{-ct})$ under [some metric].

In particular, we will obtain constants that scale like $\mathcal{O}(1/n)$, which would imply a mixing time of $\mathcal{O}(n\log n)$ [LP17].

Establishing these functional inequalities is generally not straightforward, especially for intricate probability measures like the Ising model, which exhibits various phase transitions. Conversely, being able to establish such functional inequalities illuminates physical properties of the equilibrium measure, so mixing times of Glauber (and other) dynamics have been of great interest to physicists, mathematicians, and theoretical computer scientists alike.

The approach we will take is one that is now widely used, called *measure decomposition*. The idea is to find a representation of μ in terms of simpler parts where functional inequalities (or more generally, some desired property) hold. It follows the general recipe:

> (1) find a measure decomposition, that is, a collection $(\mu_t)_{t\in\mathcal{T}}$ and $\pi \in \mathcal{P}(\mathcal{T})$ such that $\mathbb{E}_\pi[\mu_t] = \mu$;
> (2) show that the functional inequality holds for each μ_t;
> (3) show that Dirichlet form is conserved, i.e., $\mathbb{E}_\pi[\mathcal{E}_{\mu_t}] \lesssim \mathcal{E}_\mu$;
> (4) show that functional is conserved, i.e., $\text{functional}_\mu \lesssim \mathbb{E}_\pi[\text{functional}_{\mu_t}]$.

Upon following these steps, one would obtain the line of inequalities:

$$\text{functional}_\mu(f) \lesssim \mathbb{E}_\pi[\text{functional}_{\mu_t}(f)] \lesssim \mathbb{E}_\pi[\mathcal{E}_{\mu_t}(f)] \lesssim \mathcal{E}_\mu(f)$$

and the desired functional inequality is shown. The art remains in finding the right decomposition.

Remark 0.6. By the concavity of the map

$$(a,b) \mapsto \frac{ab}{a+b} \text{ on the set } \{(a,b)\in\mathbb{R}^2 : a>0, b>0\},$$

Item 3 of the recipe is always true for the case of Glauber dynamics [ALG24]. Indeed, for any f, we get from Jensen that

$$\mathbb{E}_\pi[\mathcal{E}_{\mu_t}(f)] = \frac{1}{2}\sum_{i=1}^{n}\sum_{x\in\mathcal{X}_n}\mathbb{E}_\pi\left[\frac{\mu_t(x)\mu_t(x^i)}{\mu_t(x)+\mu_t(x^i)}\right](f(x^i)-f(x))^2 \leq \mathcal{E}_\mu(f).$$

1. Nothing is easier than product measures

Product measures are usually desired because many functional inequalities are automatically satisfied when the component measures satisfy the respective functional inequality. The focus of this section will be log-Sobolev inequalities, which relate the Dirichlet form to the entropy.

Definition 1.1. For a measure μ and non-negative functional f, define the *entropy functional* to be

$$\operatorname{Ent}_\mu(f) \coloneqq \mathbb{E}_\mu[f\log f] - \mathbb{E}_\mu[f]\log\mathbb{E}_\mu[f] \geq 0.$$

We say μ satisfies a *log-Sobolev inequality* (LSI) if there exists a constant $c > 0$ such that

$$\mathcal{E}_\mu(\log f, f) \geq c\operatorname{Ent}_\mu(f).$$

The goal is to prove the following inequality.

Theorem 1.2 ([BB19, Theorem 1] or [BBD24, Theorem 6.11])**.** *Let μ be a Ising model with coupling matrix J and external field h. Moreover, let $\|J\| < 1$ and positive definite.*[1] *Then, μ satisfies a* LSI*: for all $f : \mathcal{X}_n \to \mathbb{R}$*

$$\mathrm{Ent}_\mu(f) \leq \left(1 + \frac{2\|J\|}{1 - \|J\|}\right) \mathcal{E}_\mu(\log f, f).$$

Proof. The trick here is to use that sums of Gaussians are Gaussians and quadratic interactions (though on the hypercube) structurally looks like Gaussians. For safety sake, consider the coupling matrix $J + \delta\mathbb{I}$ instead for some small $\delta > 0$—small enough so that $\alpha := \|J\| + \delta < 1$. Then, there is always a matrix C such that

$$J^{-1} = \alpha^{-1}\mathbb{I} + C^{-1},$$

which means one can decompose μ to be

$$e^{\langle x, Jx\rangle} \propto \int_{\mathbb{R}^n} e^{-\alpha^{-1}|\varphi - x|^2/2} e^{-\langle \varphi, C\varphi\rangle/2} d\varphi. \tag{1}$$

In particular, we will let the renormalized potential be

$$V(\varphi) = -\log \sum_{x \in \mathcal{X}_n} \exp\left(-\frac{1}{2\alpha}|x - \varphi|^2 + \langle h, x\rangle\right)$$

and define (respectively) the fluctuation and renormalization measure to be

$$\mu_\varphi(x) = \exp\left(-\frac{1}{2\alpha}|x - \varphi|^2 + V(\varphi)\right) \quad \text{and} \quad \pi(d\varphi) \propto e^{-V(\varphi) - \frac{1}{2}\langle \varphi, C\varphi\rangle}$$

so that in light of (1) we have

$$\mathbb{E}_\pi[\mu_\varphi] = \mu.$$

Remark 1.3. The terms *fluctuation* and *renormalization* measures are adapted from the renormalization theory by Bauerschmidt, Bondieau, and Dagallier [BBD24]. This general framework proposes to understand multiscale Gibbs measures by blurring spatial interactions by Gaussians and uses this to obtain functional inequalities independent of the scale parameter. We will respect this terminology for this subsection but such phrasing is not generic to measure decomposition, which is a far broader and vaguer set of ideas.

Notice that, for a fixed renormalized potential, μ_ϕ is a product measure. Even better, each component takes values in $\{\pm 1\}$, and for such simple measures, we can obtain very generic LSIs.

Lemma 1.4. *For each $\varphi \in \mathbb{R}^n$, μ_φ satisfies an* LSI*:*

$$\mathrm{Ent}_{\mu_\varphi}(f) \leq \frac{1}{4}\mathcal{E}_{\mu_\varphi}(\log f, f).$$

Proof. First, we reduce the problem from product measures to individual components through *tensorization*: if two measure ν_1 and ν_2 satisfy LSIs with constants c_1 and c_2 respectively, then a LSI holds for $\nu_1 \otimes \nu_2$ with constant $c_1 \vee c_2$. Then, by [BBD24, Proposition 6.9], Bernoulli random variables satisfy a LSI with constant 4. Since the constant is uniform in φ, we obtain the desired inequality. □

This completes Item (2) of the recipe. Lastly, we need to show Item (4), which requires us to look closer into how entropy interacts with conditionals. By the tower property, we see that

$$\mathrm{Ent}_\mu(f) = \mathbb{E}_\pi[\mathrm{Ent}_{\mu_\varphi}(f)] + \mathrm{Ent}_\pi(\mathbb{E}_{\mu_\varphi}[f]).$$

The first term is handled using Lemma 1.4 and concavity of the Glauber Dirichlet form (Remark 0.6).

The second term is slightly more involved in the sense that it involves log-Sobolev inequalities with a different Dirichlet form; afterall, φ is a continuous variable. The good news is the potential V is strictly convex: indeed, differentiating twice gives

$$\langle \psi, \nabla_\varphi^2 V(\varphi)\psi\rangle \geq (\alpha - \alpha^2)|\psi|^2.$$

From which, *Bakry-Emery* theory gives a log-Sobolev inequality with respect to the Dirichlet form for Langevin dynamics. To limit scope, we will not pursue the details here. The resulting log-Sobolev constant

[1] Norm here refers to the $2 \to 2$ operator norm, or the spectral radius of the matrix.

will be non-singular with respect to the regularization parameter $\delta > 0$. Sending $\delta \to 0$ completes the proof. □

As a corollary, we obtain fast mixing for several well-known spin-glass models by examining the spectrum of the coupling matrix. We start with the popular Sherrington-Kirkpatrick model, which is a mean-field model of the Edwards-Anderson model.

Definition 1.5. Let $\mathsf{G} \in \mathbb{R}^{n\times n}$ be a random matrix where $\mathbf{G}_{i,j} \sim \mathsf{N}(0, \beta^2/n)$ independently for some $\beta > 0$. The *Sherrington-Kirkpatrick* (SK) model is an (random) Ising model with $\mathbf{G}$ as coupling matrix and no external field.

Corollary 1.6. *For $\beta < 1/4$, Glauber dynamics on the* SK *model (with high probability) mixes quickly (in entropy).*

Proof. Random matrix theory tells us that, without β scaling, the spectral radius of the random matrix of interest is bounded by 2 with high probability. However, $\mathbf{G}$ is not positive definite a priori, so we shift G to move the spectrum, which now has spectrum on the interval $[0, 4]$. Therefore, we pick $\beta < 1/4$ and appeal to Theorem 1.2 to conclude an LSI inequality holds. □

Another spin-glass model of our particular interest is the so-called diluted spin-glass.

Definition 1.7. Fix $d \geq 2$ and let J be the adjacency matrix of a d-regular graph with entries $\pm\beta$ picked uniformly at random. The *d-regular diluated spin-glass* model is a (random) Ising model with coupling J and no external field.

Corollary 1.8. *For $\beta < 1/4\sqrt{d-1}$, Glauber dynamics on the d-regular diluted spin-glass model mixes quickly (in entropy).*

Proof. Again, appealing to spectral theory, $\|J\| \leq 2\beta\sqrt{d-1}$ with high probability. The rest follows from the same argument as before. □

2. Finding rank-one matrices through martingales

Product measures are nice, but to obtain the original measure as a mixture of product measures can be difficult. However, step (2) of the recipe only requires that the components be simple enough to obtain the desired functional inequality, so we can use other people's results for step (2) provided that we can construct the decomposition.

The goal for this subsection is to sketch the proof for the following result.

Theorem 2.1 ([EKZ22, Theorem 1])**.** *Let $0 \preceq J \preceq \mathbb{I}_n$, $h \in \mathbb{R}^n$, and let μ be the Gibbs measure from the Ising model with parameters (h, J). Then, for any $f : \{\pm1\}^n \to \mathbb{R}$, μ satisfies a Poincaré inequality*

$$(1 - \|J\|)\mathrm{Var}_\mu[f] \leq \mathcal{E}_\mu(f).$$

The idea is to obtain a decomposition $\mu = \mathbb{E}[\mu_\tau]$ such that μ_τ is a rank-one matrix. The reason for wanting to decompose the measure into rank-one components is because of the following Poincaré inequality.

Lemma 2.2 ([EKZ22, Lemma 8])**.** *Let $J = uu^\intercal \in \mathbb{R}^{n\times n}$ be a rank-one matrix, define $\mu \in \mathcal{P}(\{\pm1\}^n)$ to be*

$$\mu(dx) \propto \exp\left(\frac{1}{2}\langle x, u\rangle^2 + \langle h, x\rangle\right)\lambda(dx)$$

for any $h \in \mathbb{R}^n$. Moreover, let $\mathcal{E}_\mu$ be the Dirichlet form with respect to Glauber dynamics. Then, μ satisfies a Poincaé inequality: for any $f : \{\pm1\}^n \to \mathbb{R}$,

$$(1 - |u|^2)\mathrm{Var}(f) \leq \mathcal{E}_\mu(f).$$

Proof. The proof uses a result of [Wu06] that relates the *influence matrix* obtained from random vectors on the hypercube to Poincaré inequalities. Specifically, the influence matrix A of a random vector $X \sim \mu$ is one define component-wise by

$$A_{ij} \coloneqq \max_{x_{-i}, x'_{-i}} d_{\mathsf{TV}}\left(\mu(x_i|x_{-i}), \mu(x_i|x'_{-i})\right)$$

for x_{-i} and x'_{-i} differs only at the j-th entry; the diagonal $A_{ii} = 0$. [Wu06, Theorem 2.1] states that the Dirichlet form for Glauber dynamics generically satisfies a Poincaré inequality with constant $1 - \|A\|$. Restricting ourselves to rank-one models, one can derive bounds on the influence matrix. □

The main task now is to construct a decomposition of μ using rank-one matrices. Since there is not a canonical rank-one decomposition, we will try to obtain the decomposition by equipping μ with some dynamics and stopping it appropriately.

Definition 2.3. Let $\mu \in \mathcal{P}(\mathcal{X}_n)$. A *stochastic localization* process $(\mu_t)_{t\geq 0}$ with driving matrix C_t is a process defined by the Radon-Nikodym $F_t = d\mu_t/d\mu$:

$$dF_t(x) = F_t(x)\langle C_t(x - a_t), dB_t\rangle \text{ where } a_t = \sum_{y\in\mathcal{X}_n} y\mu_t(y)$$

and B is a standard Brownian motion.

Remark 2.4. Stochastic localization is an extremely popular technique at this point. It was used initially for high-dimensional geometry, most notably, in (almost) proving the KLS conjecture [Eld13, LV17, Che21] and uncovering the deep relationship between geometry and mixing of Markov chains [CE22, AKV24].

Since integration with respect to a martingale is a (local) martingale, stochastic localization process is also a measure-valued martingale, that is, for all $f \in \mathcal{C}_b(\mathcal{X}_n; \mathbb{R})$, $(\langle f, \mu_t\rangle)_{t\geq 0}$ is a martingale:

$$\begin{aligned}\langle f, \mu_t\rangle = \int_{\mathcal{X}_n} f(x)F_t(x)\mu(dx) &= \int_{\mathcal{X}_n}\int_0^t f(x)F_s(x)\langle C_s(x - a_s), dB_s\rangle\mu(dx)\\ &= \int_0^t \left(\int_{\mathcal{X}_n} f(x)C_s(x - a_s)\mu_t(dx)\right)\cdot dB_s.\end{aligned}$$

Perhaps similarly important is the fact that $F_\cdot(x)$ is an exponential martingale. By Itô,

$$\begin{aligned}\log F_t(x) &= \int_0^t \frac{1}{F_s(x)}dF_s(x) - \frac{1}{2}\int_0^t \frac{1}{F_s(x)^2}d[F_\cdot(x)]_s\\ &= \int_0^t C_s(x - a_s)\cdot dB_s - \frac{1}{2}\int_0^t |C_s(x - a_s)|^2 ds.\end{aligned}$$

Taking an exponential on both sides yields:

$$F_t(x) = \exp\left(\int_0^t C_s(x - a_s)\cdot dB_s - \frac{1}{2}\int_0^t |C_s(x - a_s)|^2 ds\right).$$

In particular, when μ is the Ising model, we see that stochastic localization is simply changing the coupling matrix and external field by driving matrix C_t.

Remark 2.5. Taking $C_t = J^{-1/2}$, then we can see that

$$\mu_t(x) \propto \exp\left(\langle x, (1-t)Jx\rangle + \langle q_t, x\rangle\right)$$

and we get a product measure at $t = 1$. This approach is much more similar to the previous approach in [BB19]. Indeed, [AKV24] uses this approach to establish *approximate tensorization* for high-temperature Ising models, which implies both log-Sobolev and Poincaré inequalities.

In this case, let J_t be the coupling matrix along μ_t and define

$$\tau = \inf\left\{t > 0 : \text{rank}J_t = 1\right\}.$$

Then, by the fact that μ_t is a martingale, optional stopping theorem gives

$$\mu = \mathbb{E}[\mu_\tau]$$

and we've obtain a decomposition (Item 1 and 2 of the recipe). Our job now is to make sure we can design a driving matrix C_t so that Item 4 of the recipe is satisfied, i.e., the variance is approximately conserved at the stopping time. And notice this is non-trivial as variance is a convex function, so variance along μ_t generally grows (variance is a supermartingale)!

Let's examine the variance more carefully. Fix a text function $f : \{\pm 1\}^n \to \mathbb{R}$ and let's examine the evolution of the variance functional under stochastic localization. For any $f : \mathcal{X}_n \to \mathbb{R}$, define $M_t = \mathbb{E}_{\mu_t}[f]$ and $Y_t = \mathrm{Var}_{\mu_t}[f]$. Then,

$$\begin{aligned} dY_t &= d\left(\int_{\mathcal{X}_n} f^2 d\mu_t - M_t^2\right) \\ &= \left\langle C_t \int_{\mathcal{X}_n} f(x)^2(x-a_t)\mu_t(dx), dB_t\right\rangle - 2\left\langle C_t \int_{\mathcal{X}_n} f(x)(x-a_t)\mu_t(dx), dB_t\right\rangle - d[M]_t. \end{aligned}$$

Therefore, if we can choose C_t so that

(1) $[M]$ is controlled up to the stopping time τ, e.g.,

$$|\mathbb{E}[Y_\tau - Y_0]| \le \delta^2\, \mathbb{E}[\tau]$$

for any δ (where the choice of C_t can depend on), and

(2) $\mathbb{E}[\tau]$ and $\|J\|_\tau$ are bounded above (by c_τ and c_J, respectively) independently of δ,

then we will get conservation of variance because:

$$\mathrm{Var}_\mu[f] \le \mathbb{E}[\mathrm{Var}_{\mu_\tau}[f] + \delta^2\tau].$$

Sending $\delta \to 0$ would complete the proof. Of course, this is highly non-trivial and I defer any enthusiasm to the paper itself [EKZ22].

We conclude with corollaries on spin-glass models as well as a few remarks.

Corollary 2.6. *For $\beta < 1/4$, Glauber dynamics on the* SK *model (with high probability) mixes quickly (in variance).*

Corollary 2.7. *For $\beta < 1/4\sqrt{d-1}$, Glauber dynamics on the d-regular diluted spin-glass model mixes quickly (in variance).*

Remark 2.8. The renormalization techniques used before, in their generality, can also be viewed from a stochastic localization perspective [BBD24, Section 5]. More specifically, stochastic localization paths can be viewed as the stochastic characteristics of the renormalization flow introduced.

Remark 2.9. Here, stochastic localization was used as a proof technique for obtaining functional inequalities. Recently, it was discovered that it can be used as an algorithm for sampling from mean-field spin-glasses [AMS22, AMS24, HMP24], though some of these results have recently been proved to be obsolete due to more ingenious bounds [HMRW24]. The algorithmic side also has connections to denoising diffusion models [Mon23].

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